XXXV Asian Pacific Mathematics Olympiad

March, 2023

Time allowed: 4 hours *Each problem is worth 7 points*

The contest problems are to be kept confidential until they are posted on the official APMO website `http://apmo-official.org`*. Please do not disclose nor discuss the problems online until that date. The use of calculators is not allowed.*

Problem 1. Let $n \geq 5$ be an integer. Consider n squares with side lengths $1, 2, \ldots, n$, respectively. The squares are arranged in the plane with their sides parallel to the x and y axes. Suppose that no two squares touch, except possibly at their vertices.
Show that it is possible to arrange these squares in a way such that every square touches exactly two other squares.

Problem 2. Find all integers n satisfying $n \geq 2$ and $\frac{\sigma(n)}{p(n)-1} = n$, in which $\sigma(n)$ denotes the sum of all positive divisors of n, and $p(n)$ denotes the largest prime divisor of n.

Problem 3. Let $ABCD$ be a parallelogram. Let W, X, Y, and Z be points on sides AB, BC, CD, and DA, respectively, such that the incenters of triangles AWZ, BXW, CYX and DZY form a parallelogram. Prove that $WXYZ$ is a parallelogram.

Problem 4. Let $c > 0$ be a given positive real and $\mathbb{R}_{>0}$ be the set of all positive reals. Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f((c+1)x + f(y)) = f(x + 2y) + 2cx \quad \text{for all } x, y \in \mathbb{R}_{>0}.$$

Problem 5. There are n line segments on the plane, no three intersecting at a point, and each pair intersecting once in their respective interiors. Tony and his $2n-1$ friends each stand at a distinct endpoint of a line segment. Tony wishes to send Christmas presents to each of his friends as follows:
First, he chooses an endpoint of each segment as a "sink". Then he places the present at the endpoint of the segment he is at. The present moves as follows:

- If it is on a line segment, it moves towards the sink.
- When it reaches an intersection of two segments, it changes the line segment it travels on and starts moving towards the new sink.

If the present reaches an endpoint, the friend on that endpoint can receive their present. Prove that Tony can send presents to exactly n of his $2n-1$ friends.