# XXXVI Asian Pacific Mathematics Olympiad

**March, 2024**

*Time allowed: 4 hours* *Each problem is worth 7 points*

*The contest problems are to be kept confidential until they are posted on the official APMO website* `http://apmo-official.org`*. Please do not disclose nor discuss the problems online until that date. The use of calculators is not allowed.*

**Problem 1.** Let $ABC$ be an acute triangle. Let $D$ be a point on side $AB$ and $E$ be a point on side $AC$ such that lines $BC$ and $DE$ are parallel. Let $X$ be an interior point of $BCED$. Suppose rays $DX$ and $EX$ meet side $BC$ at points $P$ and $Q$, respectively such that both $P$ and $Q$ lie between $B$ and $C$. Suppose that the circumcircles of triangles $BQX$ and $CPX$ intersect at a point $Y \neq X$. Prove that points $A$, $X$, and $Y$ are collinear.

**Problem 2.** Consider a $100 \times 100$ table, and identify the cell in row $a$ and column $b$, $1 \leq a, b \leq 100$, with the ordered pair $(a, b)$. Let $k$ be an integer such that $51 \leq k \leq 99$. A *$k$-knight* is a piece that moves one cell vertically or horizontally and $k$ cells to the other direction; that is, it moves from $(a, b)$ to $(c, d)$ such that $(|a-c|, |b-d|)$ is either $(1, k)$ or $(k, 1)$. The $k$-knight starts at cell $(1, 1)$, and performs several moves. A *sequence of moves* is a sequence of cells $(x_0, y_0) = (1, 1)$, $(x_1, y_1)$, $(x_2, y_2)$, $\ldots$, $(x_n, y_n)$ such that, for all $i = 1, 2, \ldots, n$, $1 \leq x_i, y_i \leq 100$ and the $k$-knight can move from $(x_{i-1}, y_{i-1})$ to $(x_i, y_i)$. In this case, each cell $(x_i, y_i)$ is said to be *reachable*. For each $k$, find $L(k)$, the number of reachable cells.

**Problem 3.** Let $n$ be a positive integer and $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\sum_{i=1}^{n} \frac{1}{2^i} \left( \frac{2}{1+a_i} \right)^{2^i} \geq \frac{2}{1 + a_1 a_2 \ldots a_n} - \frac{1}{2^n}.$$

**Problem 4.** Prove that for every positive integer $t$ there is a unique permutation $a_0, a_1, \ldots, a_{t-1}$ of $0, 1, \ldots, t-1$ such that, for every $0 \leq i \leq t-1$, the binomial coefficient $\binom{t+i}{2a_i}$ is odd and $2a_i \neq t + i$.

**Problem 5.** Line $\ell$ intersects sides $BC$ and $AD$ of cyclic quadrilateral $ABCD$ in its interior points $R$ and $S$ respectively, and intersects ray $DC$ beyond point $C$ at $Q$, and ray $BA$ beyond point $A$ at $P$. Circumcircles of the triangles $QCR$ and $QDS$ intersect at $N \neq Q$, while circumcircles of the triangles $PAS$ and $PBR$ intersect at $M \neq P$. Let lines $MP$ and $NQ$ meet at point $X$, lines $AB$ and $CD$ meet at point $K$ and lines $BC$ and $AD$ meet at point $L$. Prove that point $X$ lies on line $KL$.