APMO 1994 – Problems and Solutions

Problem 1
Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that

(i) For all $x, y \in \mathbb{R}$,
$$f(x) + f(y) + 1 \ge f(x+y) \ge f(x) + f(y),$$

(ii) For all $x \in [0,1)$, $f(0) \ge f(x)$,

(iii) $-f(-1) = f(1) = 1$.

Find all such functions f.

Answer: $f(x) = \lfloor x \rfloor$, the largest integer that does not exceed x, is the only function.

Solution
Plug $y \to 1$ in (i):

$$f(x) + f(1) + 1 \ge f(x+1) \ge f(x) + f(1) \iff f(x) + 1 \le f(x+1) \le f(x) + 2.$$

Now plug $y \to -1$ and $x \to x+1$ in (i):

$$f(x+1) + f(-1) + 1 \ge f(x) \ge f(x+1) + f(-1) \iff f(x) \le f(x+1) \le f(x) + 1.$$

Hence $f(x+1) = f(x) + 1$ and we only need to define $f(x)$ on $[0,1)$. Note that $f(1) = f(0) + 1 \implies f(0) = 0$.
Condition (ii) states that $f(x) \le 0$ in $[0,1)$.
Now plug $y \to 1 - x$ in (i):

$$f(x) + f(1-x) + 1 \le f(x + (1-x)) \le f(x) + f(1-x) \implies f(x) + f(1-x) \ge 0.$$

If $x \in (0,1)$ then $1 - x \in (0,1)$ as well, so $f(x) \le 0$ and $f(1-x) \le 0$, which implies $f(x) + f(1-x) \le 0$. Thus, $f(x) = f(1-x) = 0$ for $x \in (0,1)$. This combined with $f(0) = 0$ and $f(x+1) = f(x) + 1$ proves that $f(x) = \lfloor x \rfloor$, which satisfies the problem conditions, as since

$$x + y = \lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\} \text{ and } 0 \le \{x\} + \{y\} < 2 \implies \lfloor x \rfloor + \lfloor y \rfloor \le x + y < \lfloor x \rfloor + \lfloor y \rfloor + 2$$

implies
$$\lfloor x \rfloor + \lfloor y \rfloor + 1 \ge \lfloor x + y \rfloor \ge \lfloor x \rfloor + \lfloor y \rfloor.$$

Problem 2
Given a nondegenerate triangle ABC, with circumcentre O, orthocentre H, and circumradius R, prove that $|OH| < 3R$.

Solution 1
Embed ABC in the complex plane, with A, B and C in the circle $|z| = R$, so O is the origin. Represent each point by its lowercase letter. It is well known that $h = a + b + c$, so

$$OH = |a + b + c| \leq |a| + |b| + |c| = 3R.$$

The equality cannot occur because a, b, and c are not collinear, so $OH < 3R$.

Solution 2
Suppose with loss of generality that $\angle A < 90°$. Let BD be an altitude. Then

$$AH = \frac{AD}{\cos(90° - C)} = \frac{AB \cos A}{\sin C} = 2R \cos A.$$

By the triangle inequality,

$$OH < AO + AH = R + 2R \cos A < 3R.$$

Comment: With a bit more work, if a, b, c are the sidelengths of ABC, one can show that

$$OH^2 = 9R^2 - a^2 - b^2 - c^2.$$

In fact, using vectors in a coordinate system with O as origin, by the Euler line

$$\overrightarrow{OH} = 3\overrightarrow{OG} = 3 \cdot \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

so

$$OH^2 = \overrightarrow{OH} \cdot \overrightarrow{OH} = (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) \cdot (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$$

Expanding and using the fact that $\overrightarrow{OX} \cdot \overrightarrow{OX} = OX^2 = R^2$ for $X \in \{A, B, C\}$, as well as

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = OA \cdot OB \cdot \cos \angle AOB = R^2 \cos 2C = R^2(1 - 2\sin^2 C) = R^2 \left(1 - 2\left(\frac{c}{2R}\right)^2\right) = R^2 - \frac{c^2}{2},$$

we find that

$$\begin{aligned} OH^2 &= \overrightarrow{OA} \cdot \overrightarrow{OA} + \overrightarrow{OB} \cdot \overrightarrow{OB} + \overrightarrow{OC} \cdot \overrightarrow{OC} + 2\overrightarrow{OA} \cdot \overrightarrow{OB} + 2\overrightarrow{OA} \cdot \overrightarrow{OC} + 2\overrightarrow{OB} \cdot \overrightarrow{OC} \\ &= 3R^2 + (2R^2 - c^2) + (2R^2 - b^2) + (2R^2 - a^2) \\ &= 9R^2 - a^2 - b^2 - c^2, \end{aligned}$$

as required.
This proves that $OH^2 < 9R^2 \implies OH < 3R$, and since a, b, c can be arbitrarily small (fix the circumcircle and choose A, B, C arbitrarily close in this circle), the bound is sharp.

Problem 3
Let n be an integer of the form $a^2 + b^2$, where a and b are relatively prime integers and such that if p is a prime, $p \le \sqrt{n}$, then p divides ab. Determine all such n.

Answer: $n = 2, 5, 13$.

Solution
A prime p divides ab if and only if divides either a or b. If $n = a^2 + b^2$ is a composite then it has a prime divisor $p \le \sqrt{n}$, and if p divides a it divides b and vice-versa, which is not possible because a and b are coprime. Therefore n is a prime.
Suppose without loss of generality that $a \ge b$ and consider $a-b$. Note that $a^2+b^2 = (a-b)^2+2ab$.

- If $a = b$ then $a = b = 1$ because a and b are coprime. $n = 2$ is a solution.

- If $a-b = 1$ then a and b are coprime and $a^2+b^2 = (a-b)^2+2ab = 2ab+1 = 2b(b+1)+1 = 2b^2 + 2b + 1$. So any prime factor of any number smaller than $\sqrt{2b^2 + 2b + 1}$ is a divisor of $ab = b(b + 1)$.

  One can check that $b = 1$ and $b = 2$ yields the solutions $n = 1^2 + 2^2 = 5$ (the only prime p is 2) and $n = 2^2 + 3^2 = 13$ (the only primes p are 2 and 3). Suppose that $b > 2$.

  Consider, for instance, the prime factors of $b - 1 \le \sqrt{2b^2 + 2b + 1}$, which is coprime with b. Any prime must then divide $a = b + 1$. Then it divides $(b + 1) - (b - 1) = 2$, that is, $b - 1$ can only have 2 as a prime factor, that is, $b - 1$ is a power of 2, and since $b - 1 \ge 2$, b is odd.

  Since $2b^2 + 2b + 1 - (b + 2)^2 = b^2 - 2b - 3 = (b - 3)(b + 1) \ge 0$, we can also consider any prime divisor of $b + 2$. Since b is odd, b and $b + 2$ are also coprime, so any prime divisor of $b + 2$ must divide $a = b + 1$. But $b + 1$ and $b + 2$ are also coprime, so there can be no such primes. This is a contradiction, and $b \ge 3$ does not yield any solutions.

- If $a - b > 1$, consider a prime divisor p of $a - b = \sqrt{a^2 - 2ab + b^2} < \sqrt{a^2 + b^2}$. Since p divides one of a and b, p divides both numbers (just add or subtract $a - b$ accordingly.) This is a contradiction.

Hence the only solutions are $n = 2, 5, 13$.

Problem 4
Is there an infinite set of points in the plane such that no three points are collinear, and the distance between any two points is rational?

Answer: Yes.

Solution 1
The answer is *yes* and we present the following construction: the idea is considering points in the unit circle of the form $P_n = (\cos(2n\theta), \sin(2n\theta))$ for an appropriate θ. Then the distance P_mP_n is the length of the chord with central angle $(2m-2n)\theta \bmod \pi$, that is, $2|\sin((m-n)\theta)|$. Our task is then finding θ such that (i) $\sin(k\theta)$ is rational for all $k \in \mathbb{Z}$; (ii) points P_n are all distinct. We claim that $\theta \in (0, \pi/2)$ such that $\cos\theta = \frac{3}{5}$ and therefore $\sin\theta = \frac{4}{5}$ does the job.

Proof of (i): We know that $\sin((n+1)\theta) + \sin((n-1)\theta) = 2\sin(n\theta)\cos\theta$, so if $\sin((n-1)\theta$ and $\sin(n\theta)$ are both rational then $\sin((n+1)\theta)$ also is. Since $\sin(0\theta) = 0$ and $\sin\theta$ are rational, an induction shows that $\sin(n\theta)$ is rational for $n \in \mathbb{Z}_{>0}$; the result is also true if n is negative because sin is an odd function.

Proof of (ii): $P_m = P_n \iff 2n\theta = 2m\theta + 2k\pi$ for some $k \in \mathbb{Z}$, which implies $\sin((n-m)\theta) = \sin(k\pi) = 0$. We show that $\sin(k\theta) \neq 0$ for all $k \neq 0$.
We prove a stronger result: let $\sin(k\theta) = \frac{a_k}{5^k}$. Then

$$\sin((k+1)\theta) + \sin((k-1)\theta) = 2\sin(k\theta)\cos\theta \iff \frac{a_{k+1}}{5^{k+1}} + \frac{a_{k-1}}{5^{k-1}} = 2 \cdot \frac{a_k}{5^k} \cdot \frac{3}{5}$$
$$\iff a_{k+1} = 6a_k - 25a_{k-1}.$$

Since $a_0 = 0$ and $a_1 = 4$, a_k is an integer for $k \geq 0$, and $a_{k+1} \equiv a_k \pmod 5$ for $k \geq 1$ (note that $a_{-1} = -\frac{4}{25}$ is not an integer!). Thus $a_k \equiv 4 \pmod 5$ for all $k \geq 1$, and $\sin(k\theta) = \frac{a_k}{5^k}$ is an irreducible fraction with 5^k as denominator and $a_k \equiv 4 \pmod 5$. This proves (ii) and we are done.

Solution 2
We present a different construction. Consider the (collinear) points

$$P_k = \left(1, \frac{x_k}{y_k}\right),$$

such that the distance OP_k from the origin O,

$$OP_k = \frac{\sqrt{x_k^2 + y_k^2}}{y_k},$$

is rational, and x_k and y_k are integers. Clearly, $P_iP_j = \left|\frac{x_i}{y_i} - \frac{x_j}{y_j}\right|$ is rational.
Perform an inversion with center O and unit radius. It maps the line $x = 1$, which contains all points P_k, to a circle (minus the origin). Let Q_k be the image of P_k under this inversion. Then

$$Q_iQ_j = \frac{1^2 P_iP_j}{OP_i \cdot OP_j}$$

is rational and we are done if we choose x_k and y_k accordingly. But this is not hard, as we can choose the legs of a Pythagorean triple, say

$$x_k = k^2 - 1, \quad y_k = 2k.$$

This implies $OP_k = \frac{k^2+1}{2k}$, and then

$$Q_iQ_j = \frac{\left|\frac{i^2-1}{i} - \frac{j^2-1}{j}\right|}{\frac{i^2+1}{2i} \cdot \frac{j^2+1}{2j}} = \frac{|4(i-j)(ij+1)|}{(i^2+1)(j^2+1)}.$$

Problem 5

You are given three lists A, B, and C. List A contains the numbers of the form 10^k in base 10, with k any integer greater than or equal to 1. Lists B and C contain the same numbers translated into base 2 and 5 respectively:

A	B	C
10	1010	20
100	1100100	400
1000	1111101000	13000
$\vdots$	$\vdots$	$\vdots$

Prove that for every integer $n > 1$, there is exactly one number in exactly one of the lists B or C that has exactly n digits.

Solution

Let b_k and c_k be the number of digits in the kth term in lists B and C, respectively. Then

$$2^{b_k-1} \le 10^k < 2^{b_k} \iff \log_2 10^k < b_k \le \log_2 10^k + 1 \iff b_k = \lfloor k \cdot \log_2 10 \rfloor + 1$$

and, similarly

$$c_k = \lfloor k \cdot \log_5 10 \rfloor + 1.$$

Beatty's theorem states that if α and β are irrational positive numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

then the sequences $\lfloor k\alpha \rfloor$ and $\lfloor k\beta \rfloor$, $k = 1, 2, \ldots$, partition the positive integers.
Then, since

$$\frac{1}{\log_2 10} + \frac{1}{\log_5 10} = \log_{10} 2 + \log_{10} 5 = \log_{10}(2 \cdot 5) = 1,$$

the sequences $b_k - 1$ and $c_k - 1$ partition the positive integers, and therefore each integer greater than 1 appears in b_k or c_k exactly once. We are done.

Comment: For the sake of completeness, a proof of Beatty's theorem follows.
Let $x_n = \alpha n$ and $y_n = \beta n$, $n \ge 1$ integer. Note that, since $\alpha m = \beta n$ implies that $\frac{\alpha}{\beta}$ is rational but

$$\frac{\alpha}{\beta} = \alpha \cdot \frac{1}{\beta} = \alpha \left(1 - \frac{1}{\alpha}\right) = \alpha - 1$$

is irrational, the sequences have no common terms, and all terms in both sequences are irrational.
The theorem is equivalent to proving that exactly one term of either x_n of y_n lies in the interval $(N, N+1)$ for each N positive integer. For that purpose we count the number of terms of the union of the two sequences in the interval $(0, N)$: since $n\alpha < N \iff n < \frac{N}{\alpha}$, there are $\left\lfloor \frac{N}{\alpha} \right\rfloor$ terms of x_n in the interval and, similarly, $\left\lfloor \frac{N}{\beta} \right\rfloor$ terms of y_n in the same interval. Since the sequences are disjoint, the total of numbers is

$$T(N) = \left\lfloor \frac{N}{\alpha} \right\rfloor + \left\lfloor \frac{N}{\beta} \right\rfloor.$$

However, $x - 1 < \lfloor x \rfloor < x$ for nonintegers x, so

$$\frac{N}{\alpha} - 1 + \frac{N}{\beta} - 1 < T(N) < \frac{N}{\alpha} + \frac{N}{\beta} \iff N\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) - 2 < T(N) < N\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$
$$\iff N - 2 < T(N) < N,$$

that is, $T(N) = N - 1$.
Therefore the number of terms in $(N, N+1)$ is $T(N+1) - T(N) = N - (N-1) = 1$, and the result follows.