# Solutions

**Problem 1.** Let M'N' and P'Q' be two segments perpendicular to BD at a distance d. We need to prove that the perimeters AMNCQP and AM'N'CQ'P' are equal. Denote by S' the projection of N into M'N' and by S the projection of P' into PQ. The triangles NS'N' and P'SP are equal. (right triangles such that NS' = P'S and equal angles $\angle$P'PS = $\angle$NN'S'.) So PP' = NN'. Since it is clear that MM' = NN' , we have MM' = NN' = PP' = QQ'. On the other hand, since SP = S'N', using T' and T the projections of M and Q' into M'N' and PQ, respectively, we have

T'M' = S'N' = PS = TQ.

So P'Q' + M'N' = ST + M'T' + T'S' + S'N'
= ST + PS + TQ + MN
= PQ + MN.

(Without loss of generality M' lies between M and A)
So the perimeter of AMNCQP is:

AM + MN + NC + CQ + QP + PA =
AM' + M'M + MN + NN' + N'C + CQ + QP + PA =
AM' + PP' + MN + QQ' + N'C + CQ + QP + PA =
AM' + MN + QP + PP' + PA + QQ' + CQ + N'C =
AM' + MN + QT + TS + SP + PP' + PA + QQ' + CQ + N'C =
AM' + M'T '+ T'S' + S'N' + N'C + CQ + QQ' + Q'P' + P'P + PA =
AM' + M'N' + N'C + CQ' + Q'P' + P'A,

which is the perimeter of AM'N'CQ'P'.

Points to be given for showing that:

| | |
|---|---|
| MM' = NN' = PP' = QQ' | 1 point |
| T'M' = S'N' = PS = TQ. | 1 point |
| P'Q' + M'N' = PQ + MN. | 2 points |
| The perimeter of AMNCQP isthe same as the perimeter of AM'N'CQ'P' | 3 points |

**Problem 2.** We first prove by induction on n that:

$$\frac{(m+n)!}{(m-n)!} = \prod_{i=1}^{n} (m^2 + m - i^2 + i).$$

1. $\dfrac{(m+1)!}{(m-1)!} = m(m+1) = m^2 + m.$

2. $\dfrac{(m+n+1)!}{(m-n-1)!} = \left(\prod_{i=1}^{n} (m^2 + m - i^2 + i)\right)(m+n+1)(m-n)$ (by induction)

$$= \left(\prod_{i=1}^{n} (m^2 + m - i^2 + i)\right)(m^2 + m - n^2 - n)$$

$$= \prod_{i=1}^{n+1} (m^2 + m - i^2 + i)$$

But $m^2 + m \geq m^2 + m - i^2 + i \geq i^2 + i - i^2 + i = 2i$, for $i \geq m$.
Therefore

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2 + m)^n.$$

Points to be given for showing that:

| | |
|---|---|
| the innequlities hold for special values | up to 1 point |
| $\frac{(m+n)!}{(m-n)!} = \prod_{i=1}^{n} (m^2 + m - i^2 + i)$ for all n | 4 points |
| $m^2 + m \geq m^2 + m - i^2 + i \geq i^2 + i - i^2 + i = 2i$, for $i \geq m$, and therefore the innequalities hold | 2 points |

**Problem 3.** Let C be the given circle. Draw four circles $C_{12}$, $C_{23}$, $C_{34}$, $C_{41}$ with centers $O_{12}$, $O_{23}$, $O_{34}$, $O_{41}$, respectively, on the circle C such that $C_{12}$ passes through $P_1$ and $P_2$, $C_{23}$ passes through $P_2$ and $P_3$, $C_{34}$ passes through $P_3$ and $P_4$, $C_{41}$ passes through $P_4$ and $P_1$. Let the other point of intersection of $C_{12}$ and $C_{23}$ be $Q_4$, the other point of intersection of $C_{23}$ and $C_{34}$ be $Q_1$, the other point of intersection of $C_{34}$ and $C_{41}$ be $Q_2$, and the other point of intersection of $C_{41}$ and $C_{12}$ be $Q_3$. Then

$$\angle Q_4P_1P_2 = \frac{1}{2}\angle Q_4O_{12}P_2 \text{ and } \angle O_{23}P_1P_2 = \frac{1}{2}\angle P_3O_{12}P_2.$$

Clearly, $O_{23}$, $Q_4$ and $P_1$ are collinear.
It follows that $\angle Q_4Q_3P_2 = \angle Q_4P_1P_2 = \angle O_{23}P_1P_2\ \angle O_{23}O_{41}P_2$. Since also $O_{41}$, $Q_3$ and $P_2$ are collinear, it follows that $Q_3Q_4$ and $O_{41}O_{23}$ are parallel.
Since $O_{ij}$ bisects The arcs $P_iP_j$ , for (i, j) = (1, 2), (2, 3), (3, 4), (4, 1) we conclude that $O_{41}\ O_{23}$ and $O_{12}\ O_{34}$ are perpendicular, and hence $Q_3\ Q_4$ and $O_{12}\ O_{34}$ are perpendicular.
Since both ($O_{12}$, $Q_3$, $P_4$) and ($O_{12}$, $Q_4$, $P_3$) are collinear triples of points, we have $\angle P_4O_{12}P_3 = \angle Q_3O_{12}Q_4$, and this angle is bisected by $O_{12}\ O_{34}$ .
Thus $Q_3$ and $Q_4$ are reflections through the axis $O_{12}O_{34}$, and so are, by a similar argument $Q_1$ and $Q_2$.
We have thus shown: $Q_1$, $Q_2$, $Q_3$, $Q_4$ form a rectangle. But as $Q_4$ lies on both the angle bisector $O_{12}P_3$ and the angle bisector $O_{23}P_1$ of the triangle $P_1P_2P_3$, the point $Q_4$ must coincide with the incenter $I_4$ of the triangle $P_1P_2P_3$, and by a similar argument, $Q_1 = I_1$, $Q_2 = I_2$ and $Q_3 = I_3$ .

Points to be given for showing that

$Q_3Q_4$ and $O_{41}O_{23}$ are parallel 2 points

$Q_3$ $Q_4$ and $O_{12}$ $O_{34}$ are perpendicular 1 points

$Q_1$, $Q_2$, $Q_3$, $Q_4$ form a rectangle 2 points

$Q_4 = I_4$ , $Q_1 = I_1$, $Q_2 = I_2$ and $Q_3 = I_3$ 1 point

**Problem 4. .** We may assume that the n couples will form x male groups and y female groups. Without loss of generality, let $x \geq y$ , and

(*) $$x + y = 17.$$

Then, by the pigeonhole theorem, there exists a male group of size $\leq \left[\frac{n}{x}\right]$ and a female group of size $\geq \left[\frac{n}{y}\right]$. By condition (2), we have

(**) $$[\tfrac{n}{y}] - [\tfrac{n}{x}] \leq 1$$

From the conditions $x + y = 17$ and $x \geq y$ follows that $x \geq 9$ , and $y \leq 8$, which in turn implies

$$[\tfrac{n}{y}] - [\tfrac{n}{x}] > [\tfrac{n}{8}] - [\tfrac{n}{9}]$$

Therefore, we only need to exclude those n such that $[\frac{n}{8}] - [\frac{n}{9}] > 1$.Let $n = 9u + s$, $0 \leq s < 9$. Then

$$[\tfrac{n}{8}] - [\tfrac{n}{9}] > 1 \Leftrightarrow [\tfrac{u+s}{8}] > 1.$$

By analyzing this condition it is clear that the only values of n that are allowed are
n = 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 27, 28, 29, 30,
31, 32, 36, 37, 38, 39, 40, 45, 46, 46, 47, 48, 54, 55, 56, 63, 63, 72.

Conversely, conditions (*) and (**) give rise to a set of discussing groups according to the following description:

Let $[\frac{n}{x}] = p$ and $[\frac{n}{y}] = q$, then $p \leq q \leq p + 1$

We have

$$n = px + \alpha, 0 \leq \alpha < x \quad \text{and } n = qy - \beta \quad 0 \leq \beta < y.$$

So we can arrange the males in $\alpha$ discussing groups of size p + 1 and x - $\alpha$ groups of p elements. The females are distributed in $\beta$ discussing groups of size q - 1, and (y - $\beta$) discussing groups of size q.

Points to be given for showing that

$x \geq 9$ , and $y \leq 8$ 1 points

one only needs to exclude those n such that $[\frac{n}{8}] - [\frac{n}{9}] > 1$ 2 points

analyzing the condition ian giving the values of n that are allowed 2 points

given p and q as described in the solution
one can arrange the males in $\alpha$ discussion groups of size p + 1
and x - $\alpha$ groups of p elements and the females in $\beta$ discussion
groups of size q - 1, and (y - $\beta$) groups of size q. 2 points

**Problem 5.** Without loss of generality we can assume that $a \geq b \geq c$.Note that if $x \geq y > 0$, then $\sqrt{y} \leq 1/2(\sqrt{x} + \sqrt{y})$, i.e.,

$$\frac{1}{2\sqrt{y}}(\sqrt{x} + \sqrt{y}) \geq 1$$

Similarly, if $y \geq x > 0$, then

$$\frac{1}{2\sqrt{y}}(\sqrt{x} + \sqrt{y}) \leq 1$$

Multiplying both inequalities by $\sqrt{x} - \sqrt{y}$ we obtain

$$\sqrt{x} - \sqrt{y} \leq \frac{1}{2\sqrt{y}}(x - y),$$

for every x, y > 0. Moreover, it is easily seen that equality occurs if and only if x = y. By applying this last inequality we obtain

$$\sqrt{a + b - c} - \sqrt{a} \leq \frac{1}{2\sqrt{a}}(b - c)$$

(**) $$\sqrt{c+a-b} - \sqrt{b} \le \frac{1}{2\sqrt{b}}(c+a-2b)$$

$$\sqrt{b+c-a} - \sqrt{c} \le \frac{1}{2\sqrt{c}}(b-a)$$

and by adding up the left hand and the right hand sides of these inequalities we have

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} - \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{a}}(b-c) + \frac{1}{\sqrt{b}}(c+a-2b) + \frac{1}{\sqrt{c}}(b-a)\right)$$

$$\frac{1}{2}\left((b-c)\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right) + (a-b)\left(\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{c}}\right)\right) \le 0,$$

i.e.,

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \le \sqrt{a} + \sqrt{b} + \sqrt{c},$$

and equality occurs if and only if the three relations in (**) are equalities, i.e., if and only if $a = b = c$.

Points to be given for showing that

$\sqrt{x} - \sqrt{y} \le \frac{1}{2\sqrt{y}}(x-y)$ 2 points

the relations in (**) hold true 1 point each

$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} - \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \le$

1 point

equality occurs if and only if the three relations in (**) are equalities, i.e., if and only $a = b = c$ 1 point