XIX Asian Pacific Mathematics Olympiad

Problem 1. Let S be a set of 9 distinct integers all of whose prime factors are at most 3. Prove that S contains 3 distinct integers such that their product is a perfect cube.

Solution. Without loss of generality, we may assume that S contains only positive integers. Let

$$S = \{2^{a_i}3^{b_i} \mid a_i, b_i \in \mathbb{Z},\ a_i, b_i \geq 0,\ 1 \leq i \leq 9\}.$$

It suffices to show that there are $1 \leq i_1, i_2, i_3 \leq 9$ such that

$$a_{i_1} + a_{i_2} + a_{i_3} \equiv b_{i_1} + b_{i_2} + b_{i_3} \equiv 0 \pmod 3. \qquad (\dagger)$$

For $n = 2^a 3^b \in S$, let's call $(a \pmod 3, b \pmod 3)$ the *type* of n. Then there are 9 possible types:

$$(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2).$$

Let $N(i,j)$ be the number of integers in S of type (i,j). We obtain 3 distinct integers whose product is a perfect cube when

(1) $N(i,j) \geq 3$ for some i, j, or

(2) $N(i,0)N(i,1)N(i,2) \neq 0$ for some $i = 0, 1, 2$, or

(3) $N(0,j)N(1,j)N(2,j) \neq 0$ for some $j = 0, 1, 2$, or

(4) $N(i_1,j_1)N(i_2,j_2)N(i_3,j_3) \neq 0$, where $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\} = \{0, 1, 2\}$.

Assume that none of the conditions (1)∼(3) holds. Since $N(i,j) \leq 2$ for all (i,j), there are at least five $N(i,j)$'s that are nonzero. Furthermore, among those nonzero $N(i,j)$'s, no three have the same i nor the same j. Using these facts, one may easily conclude that the condition (4) should hold. (For example, if one places each nonzero $N(i,j)$ in the (i,j)-th box of a regular 3×3 array of boxes whose rows and columns are indexed by 0,1 and 2, then one can always find three boxes, occupied by at least one nonzero $N(i,j)$, whose rows and columns are all distinct. This implies (4).)

Second solution. Up to (†), we do the same as above and get 9 possible types:

$$(a \pmod 3), b \pmod 3)) = (0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)$$

for $n = 2^a 3^b \in S$.

Note that (i) among any 5 integers, there exist 3 whose sum is 0 (mod 3), and that (ii) if $i, j, k \in \{0,1,2\}$, then $i+j+k \equiv 0 \pmod 3$ if and only if $i = j = k$ or $\{i,j,k\} = \{0,1,2\}$.

Let's define

T: the set of types of the integers in S;
$N(i)$: the number of integers in S of the type $(i,\cdot)$;
$M(i)$: the number of integers $j \in \{0,1,2\}$ such that $(i,j) \in T$.

If $N(i) \geq 5$ for some i, the result follows from (i). Otherwise, for some permutation (i,j,k) of $(0,1,2)$,

$$N(i) \geq 3, \quad N(j) \geq 3, \quad N(k) \geq 1.$$

If $M(i)$ or $M(j)$ is 1 or 3, the result follows from (ii). Otherwise $M(i) = M(j) = 2$. Then either

$$(i,x),(i,y),(j,x),(j,y) \in T \quad \text{or} \quad (i,x),(i,y),(j,x),(j,z) \in T$$

for some permutation (x,y,z) of $(0,1,2)$. Since $N(k) \geq 1$, at least one of $(k,x),(k,y)$ and (k,z) contained in T. Therefore, in any case, the result follows from (ii). (For example, if $(k,y) \in T$, then take $(i,y),(j,y),(k,y)$ or $(i,x),(j,z),(k,y)$ from T.)

Problem 2. Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that

$$2\angle AHI = 3\angle ABC.$$

Solution. Let D be the intersection point of the lines AH and BC. Let K be the intersection point of the circumcircle O of the triangle ABC and the line AH. Let the line through I perpendicular to BC meet BC and the minor arc BC of the circumcircle O at E and N, respectively. We have

$$\angle BIC = 180^\circ - (\angle IBC + \angle ICB) = 180^\circ - \frac{1}{2}(\angle ABC + \angle ACB) = 90^\circ + \frac{1}{2}\angle BAC = 120^\circ$$

and also $\angle BNC = 180^\circ - \angle BAC = 120^\circ = \angle BIC$. Since $IN \perp BC$, the quadrilateral $BICN$ is a kite and thus $IE = EN$.

Now, since H is the orthocenter of the triangle ABC, $HD = DK$. Also because $ED \perp IN$ and $ED \perp HK$, we conclude that $IHKN$ is an isosceles trapezoid with $IH = NK$.

Hence

$$\angle AHI = 180^\circ - \angle IHK = 180^\circ - \angle AKN = \angle ABN.$$

Since $IE = EN$ and $BE \perp IN$, the triangles IBE and NBE are congruent. Therefore

$$\angle NBE = \angle IBE = \angle IBC = \angle IBA = \frac{1}{2}\angle ABC$$

and thus

$$\angle AHI = \angle ABN = \frac{3}{2}\angle ABC.$$

Second solution. Let P, Q and R be the intersection points of BH, CH and AH with AC, AB and BC, respectively. Then we have $\angle IBH = \angle ICH$. Indeed,

$$\angle IBH = \angle ABP - \angle ABI = 30^\circ - \frac{1}{2}\angle ABC$$

and

$$\angle ICH = \angle ACI - \angle ACH = \frac{1}{2}\angle ACB - 30^\circ = 30^\circ - \frac{1}{2}\angle ABC,$$

because $\angle ABH = \angle ACH = 30^\circ$ and $\angle ACB + \angle ABC = 120^\circ$. (Note that $\angle ABP > \angle ABI$ and $\angle ACI > \angle ACH$ because AB is the longest side of the triangle ABC under the given conditions.) Therefore $BIHC$ is a cyclic quadrilateral and thus

$$\angle BHI = \angle BCI = \frac{1}{2}\angle ACB.$$

On the other hand,

$$\angle BHR = 90^\circ - \angle HBR = 90^\circ - (\angle ABC - \angle ABH) = 120^\circ - \angle ABC.$$

Therefore,

$$\begin{aligned}\angle AHI &= 180^\circ - \angle BHI - \angle BHR = 60^\circ - \frac{1}{2}\angle ACB + \angle ABC \\ &= 60^\circ - \frac{1}{2}(120^\circ - \angle ABC) + \angle ABC = \frac{3}{2}\angle ABC.\end{aligned}$$

Problem 3. Consider n disks $C_1, C_2, \ldots, C_n$ in a plane such that for each $1 \le i < n$, the center of C_i is on the circumference of C_{i+1}, and the center of C_n is on the circumference of C_1. Define the *score* of such an arrangement of n disks to be the number of pairs (i, j) for which C_i properly contains C_j. Determine the maximum possible score.

Solution. The answer is $(n-1)(n-2)/2$.

Let's call a set of n disks satisfying the given conditions an *n-configuration*. For an n-configuration $\mathcal{C} = \{C_1, \ldots, C_n\}$, let $S_{\mathcal{C}} = \{(i,j) \mid C_i \text{ properly contains } C_j\}$. So, the score of an n-configuration $\mathcal{C}$ is $|S_{\mathcal{C}}|$.

We'll show that (i) there is an n-configuration $\mathcal{C}$ for which $|S_{\mathcal{C}}| = (n-1)(n-2)/2$, and that (ii) $|S_{\mathcal{C}}| \le (n-1)(n-2)/2$ for any n-configuration $\mathcal{C}$.

Let C_1 be any disk. Then for $i = 2, \ldots, n-1$, take C_i inside C_{i-1} so that the circumference of C_i contains the center of C_{i-1}. Finally, let C_n be a disk whose center is on the circumference of C_1 and whose circumference contains the center of C_{n-1}. This gives $S_{\mathcal{C}} = \{(i,j) \mid 1 \le i < j \le n-1\}$ of size $(n-1)(n-2)/2$, which proves (i).

For any n-configuration $\mathcal{C}$, $S_{\mathcal{C}}$ must satisfy the following properties:

(1) $(i,i) \notin S_{\mathcal{C}}$,

(2) $(i+1,i) \notin S_{\mathcal{C}}$, $(1,n) \notin S_{\mathcal{C}}$,

(3) if $(i,j), (j,k) \in S_{\mathcal{C}}$, then $(i,k) \in S_{\mathcal{C}}$,

(4) if $(i,j) \in S_{\mathcal{C}}$, then $(j,i) \notin S_{\mathcal{C}}$.

Now we show that a set G of ordered pairs of integers between 1 and n, satisfying the conditions (1)$\sim$(4), can have no more than $(n-1)(n-2)/2$ elements. Suppose that there exists a set G that satisfies the conditions (1)$\sim$(4), and has more than $(n-1)(n-2)/2$ elements. Let n be the least positive integer with which there exists such a set G. Note that G must have $(i, i+1)$ for some $1 \le i \le n$ or $(n,1)$, since otherwise G can have at most

$$\binom{n}{2} - n = \frac{n(n-3)}{2} < \frac{(n-1)(n-2)}{2}$$

elements. Without loss of generality we may assume that $(n,1) \in G$. Then $(1, n-1) \notin G$, since otherwise the condition (3) yields $(n, n-1) \in G$ contradicting the condition (2). Now let $G' = \{(i,j) \in G \mid 1 \le i, j \le n-1\}$, then G' satisfies the conditions (1)$\sim$(4), with $n-1$.

We now claim that $|G - G'| \le n-2$:

Suppose that $|G - G'| > n-2$, then $|G - G'| = n-1$ and hence for each $1 \le i \le n-1$, either (i,n) or (n,i) must be in G. We already know that $(n,1) \in G$ and $(n-1,n) \in G$ (because $(n, n-1) \notin G$) and this implies that $(n, n-2) \notin G$ and $(n-2, n) \in G$. If we keep doing this process, we obtain $(1,n) \in G$, which is a contradiction.

Since $|G - G'| \leq n - 2$, we obtain

$$|G'| \geq \frac{(n-1)(n-2)}{2} - (n-2) = \frac{(n-2)(n-3)}{2}.$$

This, however, contradicts the minimality of n, and hence proves (ii).

Problem 4. Let x, y and z be positive real numbers such that $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$. Prove that

$$\frac{x^2+yz}{\sqrt{2x^2(y+z)}}+\frac{y^2+zx}{\sqrt{2y^2(z+x)}}+\frac{z^2+xy}{\sqrt{2z^2(x+y)}} \geq 1.$$

Solution. We first note that

$$\begin{aligned}
\frac{x^2+yz}{\sqrt{2x^2(y+z)}} &= \frac{x^2-x(y+z)+yz}{\sqrt{2x^2(y+z)}}+\frac{x(y+z)}{\sqrt{2x^2(y+z)}} \\
&= \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}+\sqrt{\frac{y+z}{2}} \\
&\geq \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}+\frac{\sqrt{y}+\sqrt{z}}{2}. \qquad (1)
\end{aligned}$$

Similarly, we have

$$\frac{y^2+zx}{\sqrt{2y^2(z+x)}} \geq \frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}}+\frac{\sqrt{z}+\sqrt{x}}{2}, \qquad (2)$$

$$\frac{z^2+xy}{\sqrt{2z^2(x+y)}} \geq \frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}}+\frac{\sqrt{x}+\sqrt{y}}{2}. \qquad (3)$$

We now add (1)∼(3) to get

$$\begin{aligned}
&\frac{x^2+yz}{\sqrt{2x^2(y+z)}}+\frac{y^2+zx}{\sqrt{2y^2(z+x)}}+\frac{z^2+xy}{\sqrt{2z^2(x+y)}} \\
&\geq \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}+\frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}}+\frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}}+\sqrt{x}+\sqrt{y}+\sqrt{z} \\
&= \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}+\frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}}+\frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}}+1.
\end{aligned}$$

Thus, it suffices to show that

$$\frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}+\frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}}+\frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}} \geq 0. \qquad (4)$$

Now, assume without loss of generality, that $x \geq y \geq z$. Then we have

$$\frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}} \geq 0$$

and

$$\frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}}+\frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}}=\frac{(y-z)(x-z)}{\sqrt{2z^2(x+y)}}-\frac{(y-z)(x-y)}{\sqrt{2y^2(z+x)}}$$
$$\geq \frac{(y-z)(x-y)}{\sqrt{2z^2(x+y)}}-\frac{(y-z)(x-y)}{\sqrt{2y^2(z+x)}}=(y-z)(x-y)\left(\frac{1}{\sqrt{2z^2(x+y)}}-\frac{1}{\sqrt{2y^2(z+x)}}\right).$$

The last quantity is non-negative due to the fact that

$$y^2(z+x)=y^2z+y^2x\geq yz^2+z^2x=z^2(x+y).$$

This completes the proof.

Second solution. By Cauchy-Schwarz inequality,

$$\left(\frac{x^2}{\sqrt{2x^2(y+z)}}+\frac{y^2}{\sqrt{2y^2(z+x)}}+\frac{z^2}{\sqrt{2z^2(x+y)}}\right)$$
$$\times\left(\sqrt{2(y+z)}+\sqrt{2(z+x)}+\sqrt{2(x+y)}\right)\geq(\sqrt{x}+\sqrt{y}+\sqrt{z})^2=1, \tag{5}$$

and

$$\left(\frac{yz}{\sqrt{2x^2(y+z)}}+\frac{zx}{\sqrt{2y^2(z+x)}}+\frac{xy}{\sqrt{2z^2(x+y)}}\right)$$
$$\times\left(\sqrt{2(y+z)}+\sqrt{2(z+x)}+\sqrt{2(x+y)}\right)\geq\left(\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}+\sqrt{\frac{xy}{z}}\right)^2. \tag{6}$$

We now combine (5) and (6) to find

$$\left(\frac{x^2+yz}{\sqrt{2x^2(y+z)}}+\frac{y^2+zx}{\sqrt{2y^2(z+x)}}+\frac{z^2+xy}{\sqrt{2z^2(x+y)}}\right)$$
$$\times\left(\sqrt{2(x+y)}+\sqrt{2(y+z)}+\sqrt{2(z+x)}\right)$$
$$\geq 1+\left(\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}+\sqrt{\frac{xy}{z}}\right)^2\geq 2\left(\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}+\sqrt{\frac{xy}{z}}\right).$$

Thus, it suffices to show that

$$2\left(\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}+\sqrt{\frac{xy}{z}}\right)\geq\sqrt{2(y+z)}+\sqrt{2(z+x)}+\sqrt{2(x+y)}. \tag{7}$$

Consider the following inequality using AM-GM inequality

$$\left[\sqrt{\frac{yz}{x}}+\left(\frac{1}{2}\sqrt{\frac{zx}{y}}+\frac{1}{2}\sqrt{\frac{xy}{z}}\right)\right]^2\geq 4\sqrt{\frac{yz}{x}}\left(\frac{1}{2}\sqrt{\frac{zx}{y}}+\frac{1}{2}\sqrt{\frac{xy}{z}}\right)=2(y+z),$$

or equivalently

$$\sqrt{\frac{yz}{x}} + \left(\frac{1}{2}\sqrt{\frac{zx}{y}} + \frac{1}{2}\sqrt{\frac{xy}{z}}\right) \geq \sqrt{2(y+z)}\,.$$

Similarly, we have

$$\sqrt{\frac{zx}{y}} + \left(\frac{1}{2}\sqrt{\frac{xy}{z}} + \frac{1}{2}\sqrt{\frac{yz}{x}}\right) \geq \sqrt{2(z+x)}\,,$$
$$\sqrt{\frac{xy}{z}} + \left(\frac{1}{2}\sqrt{\frac{yz}{x}} + \frac{1}{2}\sqrt{\frac{zx}{y}}\right) \geq \sqrt{2(x+y)}\,.$$

Adding the last three inequalities, we get

$$2\left(\sqrt{\frac{yz}{x}} + \sqrt{\frac{zx}{y}} + \sqrt{\frac{xy}{z}}\right) \geq \sqrt{2(y+z)} + \sqrt{2(z+x)} + \sqrt{2(x+y)}\,.$$

This completes the proof.

Problem 5. A regular (5×5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.

Solution. We assign the following *first labels* to the 25 positions of the lights:

1	1	0	1	1
0	0	0	0	0
1	1	0	1	1
0	0	0	0	0
1	1	0	1	1

For each on-off combination of lights in the array, define its *first value* to be the sum of the first labels of those positions at which the lights are switched on. It is easy to check that toggling any switch always leads to an on-off combination of lights whose first value has the same parity(the remainder when divided by 2) as that of the previous on-off combination.

The $90°$ rotation of the first labels gives us another labels (let us call it the *second labels*) which also makes the parity of the *second value*(the sum of the second labels of those positions at which the lights are switched on) invariant under toggling.

1	0	1	0	1
1	0	1	0	1
0	0	0	0	0
1	0	1	0	1
1	0	1	0	1

Since the parity of the first and the second values of the initial status is 0, after certain number of toggles the parity must remain unchanged with respect to the first labels and the second labels as well. Therefore, if exactly one light is on after some number of toggles, the label of that position must be 0 with respect to both labels. Hence according to the above pictures, the possible positions are the ones marked with $*_i$'s in the following picture:

	$*_2$		$*_1$	
		$*_0$		
	$*_3$		$*_4$	

Now we demonstrate that all five positions are possible :

Toggling the positions checked by t (the order of toggling is irrelevant) in the first picture makes the center($*_0$) the only position with light on and the second picture makes the position $*_1$ the only position with light on. The other $*_i$'s can be obtained by rotating the second picture appropriately.

			t	t
		t		
	t	t		t
t				t
t		t	t	

	t		t	
t	t		t	t
	t			
		t	t	t
			t	