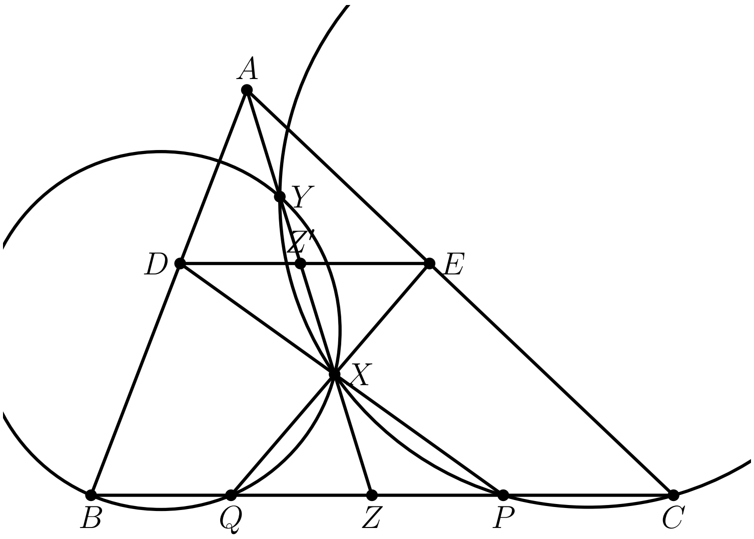
APMO 2024 – Problems and Solutions

Problem 1
Let ABC be an acute triangle. Let D be a point on side AB and E be a point on side AC such that lines BC and DE are parallel. Let X be an interior point of $BCED$. Suppose rays DX and EX meet side BC at points P and Q, respectively such that both P and Q lie between B and C. Suppose that the circumcircles of triangles BQX and CPX intersect at a point $Y \neq X$. Prove that points A, X, and Y are collinear.

Solution 1

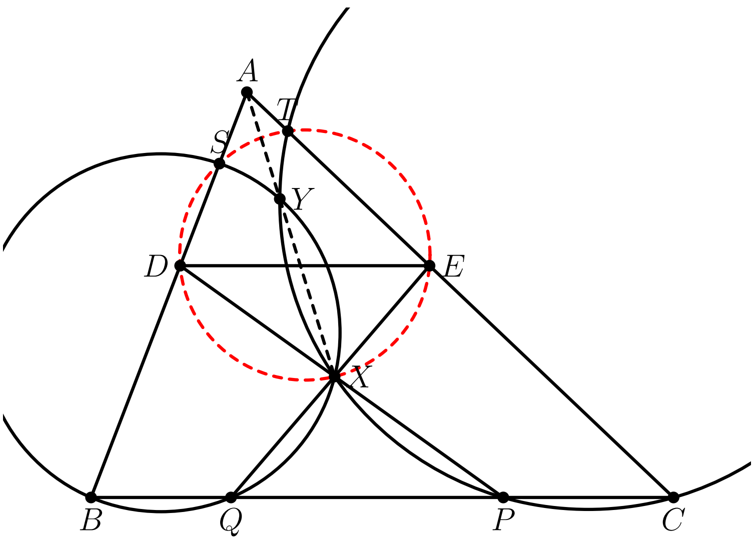


Let ℓ be the radical axis of circles BQX and CPX. Since X and Y are on ℓ, it is sufficient to show that A is on ℓ. Let line AX intersect segments BC and DE at Z and Z', respectively. Then it is sufficient to show that Z is on ℓ. By $BC \parallel DE$, we obtain

$$\frac{BZ}{ZC} = \frac{DZ'}{Z'E} = \frac{PZ}{ZQ},$$

thus $BZ \cdot QZ = CZ \cdot PZ$, which implies that Z is on ℓ.

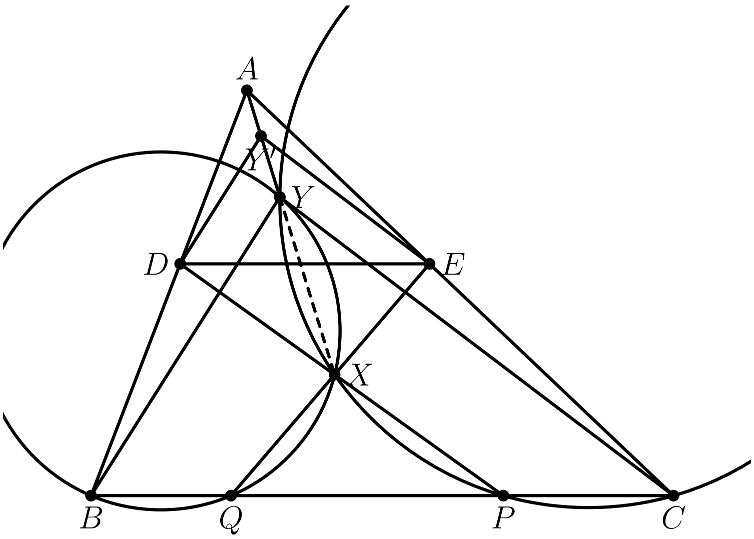
Solution 2



Let circle BQX intersect line AB at a point S which is different from B. Then $\angle DEX = \angle XQC = \angle BSX$, thus S is on circle DEX. Similarly, let circle CPX intersect line AC at a point T which is different from C. Then T is on circle DEX. The power of A with respect to the circle DEX is $AS \cdot AD = AT \cdot AE$. Since $\frac{AD}{AB} = \frac{AE}{AC}$, $AS \cdot AB = AT \cdot AC$. Then A is in the radical axis of circles BQX and CPX, which implies that three points A, X and Y are collinear.

Solution 3

Consider the (direct) homothety that takes triangle ADE to triangle ABC, and let Y' be the image of Y under this homothety; in other words, let Y' be the intersection of the line parallel to BY through D and the line parallel to CY through E.



The homothety implies that A, Y, and Y' are collinear, and that $\angle DY'E = \angle BYC$. Since $BQXY$ and $CPXY$ are cyclic,

$$\angle DY'E = \angle BYC = \angle BYX + \angle XYC = \angle XQP + \angle XPQ = 180^\circ - \angle PXQ = 180^\circ - \angle DXE,$$

which implies that $DY'EX$ is cyclic. Therefore

$$\angle DY'X = \angle DEX = \angle PQX = \angle BYX,$$

which, combined with $DY' \parallel BY$, implies $Y'X \parallel YX$. This proves that X, Y, and Y' are collinear, which in turn shows that A, X, and Y are collinear.

Problem 2

Consider a 100×100 table, and identify the cell in row a and column b, $1 \le a, b \le 100$, with the ordered pair (a, b). Let k be an integer such that $51 \le k \le 99$. A *k-knight* is a piece that moves one cell vertically or horizontally and k cells to the other direction; that is, it moves from (a, b) to (c, d) such that $(|a-c|, |b-d|)$ is either $(1, k)$ or $(k, 1)$. The k-knight starts at cell $(1, 1)$, and performs several moves. A *sequence of moves* is a sequence of cells $(x_0, y_0) = (1, 1)$, (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) such that, for all $i = 1, 2, \ldots, n$, $1 \le x_i, y_i \le 100$ and the k-knight can move from (x_{i-1}, y_{i-1}) to (x_i, y_i). In this case, each cell (x_i, y_i) is said to be *reachable.* For each k, find $L(k)$, the number of reachable cells.

Answer: $L(k) = \begin{cases} 100^2 - (2k-100)^2 & \text{if } k \text{ is even} \\ \frac{100^2-(2k-100)^2}{2} & \text{if } k \text{ is odd} \end{cases}.$

Solution

Cell (x, y) is directly reachable from another cell if and only if $x - k \ge 1$ or $x + k \le 100$ or $y - k \ge 1$ or $y + k \le 100$, that is, $x \ge k+1$ or $x \le 100-k$ or $y \ge k+1$ or $y \le 100-k$ $(*)$. Therefore the cells (x, y) for which $101 - k \le x \le k$ and $101 - k \le y \le k$ are unreachable. Let S be this set of unreachable cells in this square, namely the square of cells (x, y), $101 - k \le x, y \le k$.

If condition $(*)$ is valid for both (x, y) and $(x \pm 2, y \pm 2)$ then one can move from (x, y) to $(x \pm 2, y \pm 2)$, if they are both in the table, with two moves: either $x \le 50$ or $x \ge 51$; the same is true for y. In the first case, move $(x, y) \to (x + k, y \pm 1) \to (x, y \pm 2)$ or $(x, y) \to (x \pm 1, y + k) \to (x \pm 2, y)$. In the second case, move $(x, y) \to (x - k, y \pm 1) \to (x, y \pm 2)$ or $(x, y) \to (x \pm 1, y - k) \to (x \pm 2, y)$.

Hence if the table is colored in two colors like a chessboard, if $k \le 50$, cells with the same color as $(1, 1)$ are reachable. Moreover, if k is even, every other move changes the color of the occupied cell, and all cells are potentially reachable; otherwise, only cells with the same color as $(1, 1)$ can be visited. Therefore, if k is even then the reachable cells consists of all cells except the center square defined by $101 - k \le x \le k$ and $101 - k \le y \le k$, that is, $L(k) = 100^2 - (2k - 100)^2$; if k is odd, then only half of the cells are reachable: the ones with the same color as $(1, 1)$, and $L(k) = \frac{1}{2}(100^2 - (2k - 100)^2)$.

Problem 3
Let n be a positive integer and $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\sum_{i=1}^{n} \frac{1}{2^i}\left(\frac{2}{1+a_i}\right)^{2^i} \geq \frac{2}{1+a_1a_2\ldots a_n} - \frac{1}{2^n}.$$

Solution
We first prove the following lemma:

Lemma 1. *For k positive integer and $x, y > 0$,*

$$\left(\frac{2}{1+x}\right)^{2^k} + \left(\frac{2}{1+y}\right)^{2^k} \geq 2\left(\frac{2}{1+xy}\right)^{2^{k-1}}.$$

The proof goes by induction. For $k = 1$, we have

$$\left(\frac{2}{1+x}\right)^{2} + \left(\frac{2}{1+y}\right)^{2} \geq 2\left(\frac{2}{1+xy}\right),$$

which reduces to

$$xy(x-y)^2 + (xy-1)^2 \geq 0.$$

For $k > 1$, by the inequality $2(A^2+B^2) \geq (A+B)^2$ applied at $A = \left(\frac{2}{1+x}\right)^{2^{k-1}}$ and $B = \left(\frac{2}{1+y}\right)^{2^{k-1}}$ followed by the induction hypothesis

$$2\left(\left(\frac{2}{1+x}\right)^{2^k} + \left(\frac{2}{1+y}\right)^{2^k}\right) \geq \left(\left(\frac{2}{1+x}\right)^{2^{k-1}} + \left(\frac{2}{1+y}\right)^{2^{k-1}}\right)^2$$
$$\geq \left(2\left(\frac{2}{1+xy}\right)^{2^{k-2}}\right)^2 = 4\left(\frac{2}{1+xy}\right)^{2^{k-1}},$$

from which the lemma follows.
The problem now can be deduced from summing the following applications of the lemma, multiplied by the appropriate factor:

$$\frac{1}{2^n}\left(\frac{2}{1+a_n}\right)^{2^n} + \frac{1}{2^n}\left(\frac{2}{1+1}\right)^{2^n} \geq \frac{1}{2^{n-1}}\left(\frac{2}{1+a_n\cdot 1}\right)^{2^{n-1}}$$
$$\frac{1}{2^{n-1}}\left(\frac{2}{1+a_{n-1}}\right)^{2^{n-1}} + \frac{1}{2^{n-1}}\left(\frac{2}{1+a_n}\right)^{2^{n-1}} \geq \frac{1}{2^{n-2}}\left(\frac{2}{1+a_{n-1}a_n}\right)^{2^{n-2}}$$
$$\frac{1}{2^{n-2}}\left(\frac{2}{1+a_{n-2}}\right)^{2^{n-2}} + \frac{1}{2^{n-2}}\left(\frac{2}{1+a_{n-1}a_n}\right)^{2^{n-2}} \geq \frac{1}{2^{n-3}}\left(\frac{2}{1+a_{n-2}a_{n-1}a_n}\right)^{2^{n-3}}$$
$$\cdots$$
$$\frac{1}{2^k}\left(\frac{2}{1+a_k}\right)^{2^k} + \frac{1}{2^k}\left(\frac{2}{1+a_{k+1}\ldots a_{n-1}a_n}\right)^{2^k} \geq \frac{1}{2^{k-1}}\left(\frac{2}{1+a_k\ldots a_{n-2}a_{n-1}a_n}\right)^{2^{k-1}}$$
$$\cdots$$
$$\frac{1}{2}\left(\frac{2}{1+a_1}\right)^{2} + \frac{1}{2}\left(\frac{2}{1+a_2\ldots a_{n-1}a_n}\right)^{2} \geq \frac{2}{1+a_1\ldots a_{n-2}a_{n-1}a_n}.$$

Comment: Equality occurs if and only if $a_1 = a_2 = \cdots = a_n = 1$.

Comment: The main motivation for the lemma is trying to "telescope" the sum

$$\frac{1}{2^n} + \sum_{i=1}^{n} \frac{1}{2^i} \left(\frac{2}{1+a_i} \right)^{2^i},$$

that is,

$$\frac{1}{2} \left(\frac{2}{1+a_1} \right)^2 + \cdots + \frac{1}{2^{n-1}} \left(\frac{2}{1+a_{n-1}} \right)^{2^{n-1}} + \frac{1}{2^n} \left(\frac{2}{1+a_n} \right)^{2^n} + \frac{1}{2^n} \left(\frac{2}{1+1} \right)^{2^n}$$

to obtain an expression larger than or equal to

$$\frac{2}{1+a_1 a_2 \ldots a_n}.$$

It seems reasonable to obtain a inequality that can be applied from right to left, decreases the exponent of the factor $1/2^k$ by 1, and multiplies the variables in the denominator. Given that, the lemma is quite natural:

$$\frac{1}{2^k} \left(\frac{2}{1+x} \right)^{2^k} + \frac{1}{2^k} \left(\frac{2}{1+y} \right)^{2^k} \geq \frac{1}{2^{k-1}} \left(\frac{2}{1+xy} \right)^{2^{i-1}},$$

or

$$\left(\frac{2}{1+x} \right)^{2^k} + \left(\frac{2}{1+y} \right)^{2^k} \geq 2 \left(\frac{2}{1+xy} \right)^{2^{k-1}}.$$

Problem 4
Prove that for every positive integer t there is a unique permutation $a_0, a_1, \dots, a_{t-1}$ of $0, 1, \dots, t-1$ such that, for every $0 \le i \le t-1$, the binomial coefficient $\binom{t+i}{2a_i}$ is odd and $2a_i \ne t+i$.

Solution
We constantly make use of *Kummer's theorem* which, in particular, implies that $\binom{n}{k}$ is odd if and only if k and $n-k$ have ones in different positions in binary. In other words, if $S(x)$ is the set of positions of the digits 1 of x in binary (in which the digit multiplied by 2^i is in position i), $\binom{n}{k}$ is odd if and only if $S(k) \subseteq S(n)$. Moreover, if we set $k < n$, $S(k)$ is a proper subset of $S(n)$, that is, $|S(k)| < |S(n)|$.
We start with a lemma that guides us how the permutation should be set.

Lemma 1.
$$\sum_{i=0}^{t-1} |S(t+i)| = t + \sum_{i=0}^{t-1} |S(2i)|.$$

The proof is just realizing that $S(2i) = \{1+x, x \in S(i)\}$ and $S(2i+1) = \{0\} \cup \{1+x, x \in S(i)\}$, because $2i$ in binary is i followed by a zero and $2i+1$ in binary is i followed by a one. Therefore

$$\begin{aligned}\sum_{i=0}^{t-1} |S(t+i)| &= \sum_{i=0}^{2t-1} |S(i)| - \sum_{i=0}^{t-1} |S(i)| = \sum_{i=0}^{t-1} |S(2i)| + \sum_{i=0}^{t-1} |S(2i+1)| - \sum_{i=0}^{t-1} |S(i)| \\ &= \sum_{i=0}^{t-1} |S(i)| + \sum_{i=0}^{t-1} (1 + |S(i)|) - \sum_{i=0}^{t-1} |S(i)| = t + \sum_{i=0}^{t-1} |S(i)| = t + \sum_{i=0}^{t-1} |S(2i)|.\end{aligned}$$

The lemma has an immediate corollary: since $t+i > 2a_i$ and $\binom{t+i}{2a_i}$ is odd for all i, $0 \le i \le t-1$, $S(2a_i) \subset S(t+i)$ with $|S(2a_i)| \le |S(t+i)| - 1$. Since the sum of $|S(2a_i)|$ is t less than the sum of $|S(t+i)|$, and there are t values of i, equality must occur, that is, $|S(2a_i)| = |S(t+i)| - 1$, which in conjunction with $S(2a_i) \subset S(t+i)$ means that $t+i-2a_i = 2^{k_i}$ for every i, $0 \le i \le t-1$, $k_i \in S(t+i)$ (more precisely, $\{k_i\} = S(t+i) \setminus S(2a_i)$.)
In particular, for $t+i$ odd, this means that $t+i-2a_i = 1$, because the only odd power of 2 is 1. Then $a_i = \frac{t+i-1}{2}$ for $t+i$ odd, which takes up all the numbers greater than or equal to $\frac{t-1}{2}$. Now we need to distribute the numbers that are smaller than $\frac{t-1}{2}$ (call these numbers *small*). If $t+i$ is even then by *Lucas' Theorem* $\binom{t+i}{2a_i} \equiv \binom{\frac{t+i}{2}}{a_i} \pmod 2$, so we pair numbers from $\lceil t/2 \rceil$ to $t-1$ (call these numbers *big*) with the small numbers.
Say that a set A is *paired* with another set B whenever $|A| = |B|$ and there exists a bijection $\pi\colon A \to B$ such that $S(a) \subset S(\pi(a))$ and $|S(a)| = |S(\pi(a))| - 1$; we also say that a and $\pi(a)$ are paired. We prove by induction in t that $A_t = \{0, 1, 2, \dots, \lfloor t/2 \rfloor - 1\}$ (the set of small numbers) and $B_t = \{\lceil t/2 \rceil, \dots, t-2, t-1\}$ (the set of big numbers) can be uniquely paired.
The claim is immediate for $t = 1$ and $t = 2$. For $t > 2$, there is exactly one power of two in B_t, since $t/2 \le 2^a < t \iff a = \lceil \log_2(t/2) \rceil$. Let 2^a be this power of two. Then, since $2^a \ge t/2$, no number in A_t has a one in position a in binary. Since for every number x, $2^a \le x < t$, $a \in S(x)$ and $a \notin S(y)$ for all $y \in A_t$, x can only be paired with $x - 2^a$, since $S(x)$ needs to be stripped of exactly one position. This takes cares of $x \in B_t$, $2^a \le x < t$, and $y \in A_t$, $0 \le y < t - 2^a$.
Now we need to pair the numbers from $A' = \{t - 2^a, t - 2^a + 1, \dots, \lfloor t/2 \rfloor - 1\} \subset A$ with the numbers from $B' = \{\lceil t/2 \rceil, \lceil t/2 \rceil + 1, \dots, 2^a - 1\} \subset B$. In order to pair these $t - 2(t - 2^a) = 2^{a+1} - t < t$ numbers, we use the induction hypothesis and a bijection between $A' \cup B'$ and $B_{2^{a+1}-t} \cup A_{2^{a+1}-t}$. Let $S = S(2^a - 1) = \{0, 1, 2, \dots, a-1\}$. Then take a pair x, y, $x \in A_{2^{a+1}-t}$ and $y \in B_{2^{a+1}-t}$ and biject it with $2^a - 1 - x \in B'$ and $2^a - 1 - y \in A'$. In fact,

$$0 \le x \le \left\lfloor \frac{2^{a+1} - t}{2} \right\rfloor - 1 = 2^a - \left\lceil \frac{t}{2} \right\rceil - 1 \iff \left\lceil \frac{t}{2} \right\rceil \le 2^a - 1 - x \le 2^a - 1$$

and

$$\left\lceil \frac{2^{a+1}-t}{2} \right\rceil = 2^a - \left\lfloor \frac{t}{2} \right\rfloor \le y \le 2^{a+1}-t-1 \iff t-2^a \le 2^a-1-y \le \left\lfloor \frac{t}{2} \right\rfloor - 1.$$

Moreover, $S(2^a-1-x) = S \setminus S(x)$ and $S(2^a-1-y) = S \setminus S(y)$ are complements with respect to S, and $S(x) \subset S(y)$ and $|S(x)| = |S(y)|-1$ implies $S(2^a-1-y) \subset S(2^a-1-x)$ and $|S(2^a-1-y)| = |S(2^a-1-x)|-1$. Therefore a pairing between A' and B' corresponds to a pairing between $A_{2^{a+1}-t}$ and $B_{2^{a+1}-t}$. Since the latter pairing is unique, the former pairing is also unique, and the result follows.
We illustrate the bijection by showing the case $t = 23$:

$$A_{23} = \{0,1,2,\ldots,10\}, \quad B_{23} = \{12,13,14,\ldots,22\}.$$

The pairing is

$$\begin{pmatrix} 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 8 & 9 & 10 & 7 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix},$$

in which the bijection is between

$$\begin{pmatrix} 12 & 13 & 14 & 15 \\ 8 & 9 & 10 & 7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 2 & 1 & 0 \\ 7 & 6 & 5 & 8 \end{pmatrix} \to \begin{pmatrix} 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 0 \end{pmatrix}.$$

Problem 5

Line ℓ intersects sides BC and AD of cyclic quadrilateral $ABCD$ in its interior points R and S respectively, and intersects ray DC beyond point C at Q, and ray BA beyond point A at P. Circumcircles of the triangles QCR and QDS intersect at $N \neq Q$, while circumcircles of the triangles PAS and PBR intersect at $M \neq P$. Let lines MP and NQ meet at point X, lines AB and CD meet at point K and lines BC and AD meet at point L. Prove that point X lies on line KL.

Solution 1

We start with the following lemma.

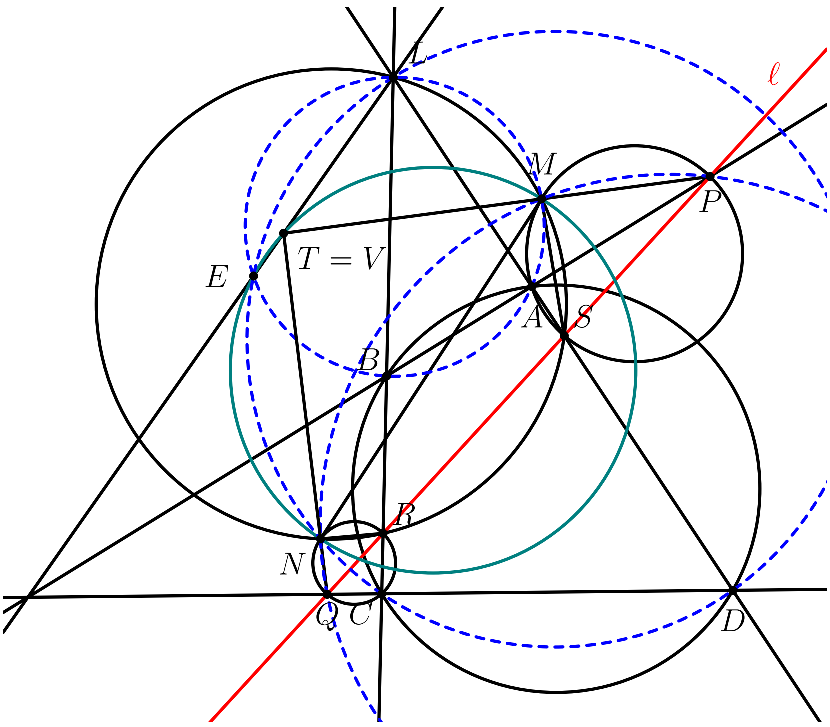
Lemma 1. *Points M, N, P, Q are concyclic.*

Point M is the Miquel point of lines $AP = AB$, $PS = \ell$, $AS = AD$, and $BR = BC$, and point N is the Miquel point of lines $CQ = CD$, $RC = BC$, $QR = \ell$, and $DS = AD$. Both points M and N are on the circumcircle of the triangle determined by the common lines AD, ℓ, and BC, which is LRS.

Then, since quadrilaterals $QNRC$, $PMAS$, and $ABCD$ are all cyclic, using directed angles (modulo 180°)

$$\begin{aligned}\measuredangle NMP &= \measuredangle NMS + \measuredangle SMP = \measuredangle NRS + \measuredangle SAP = \measuredangle NRQ + \measuredangle DAB = \measuredangle NRQ + \measuredangle DCB \\ &= \measuredangle NRQ + \measuredangle QCR = \measuredangle NRQ + \measuredangle QNR = \measuredangle NQR = \measuredangle NQP,\end{aligned}$$

which implies that $MNQP$ is a cyclic quadrilateral.



Let E be the Miquel point of $ABCD$ (that is, of lines AB, BC, CD, DA). It is well known that E lies in the line t connecting the intersections of the opposite lines of $ABCD$. Let lines NQ and t meet at T. If $T \neq E$, using directed angles, looking at the circumcircles of LAB (which contains, by definition, E and M), APS (which also contains M), and $MNQP$,

$$\measuredangle TEM = \measuredangle LEM = \measuredangle LAM = \measuredangle SAM = \measuredangle SPM = \measuredangle QPM = \measuredangle QNM = \measuredangle TNM,$$

that is, T lies in the circumcircle ω of EMN. If $T = E$, the same computation shows that $\measuredangle LEM = \measuredangle ENM$, which means that t is tangent to ω.

Now let lines MP and t meet at V. An analogous computation shows, by looking at the circumcircles of LCD (which contains E and N), CQR, and $MNQP$, that V lies in ω as well, and that if $V = E$ then t is tangent to ω.
Therefore, since ω meet t at T, V, and E, either $T = V$ if both $T \neq E$ and $V \neq E$ or $T = V = E$. At any rate, the intersection of lines MP and NQ lies in t.

Solution 2
Barycentric coordinates are a viable way to solve the problem, but even the solution we have found had some clever computations. Here is an outline of this solution.

Lemma 2. *Denote by* $\operatorname{pow}_\omega X$ *the power of point* X *with respect to circle* ω. *Let* Γ_1 *and* Γ_2 *be circles with different centers. Considering* ABC *as the reference triangle in barycentric coordinates, the radical axis of* Γ_1 *and* Γ_2 *is given by*

$$(\operatorname{pow}_{\Gamma_1} A - \operatorname{pow}_{\Gamma_2} A)x + (\operatorname{pow}_{\Gamma_1} B - \operatorname{pow}_{\Gamma_2} B)y + (\operatorname{pow}_{\Gamma_1} C - \operatorname{pow}_{\Gamma_2} C)z = 0.$$

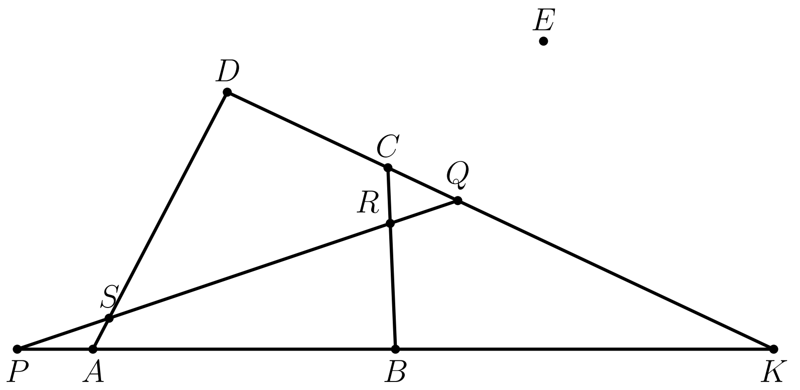
Proof: Let Γ_i have the equation $\Gamma_i(x, y, z) = -a^2yz - b^2zx - c^2xy + (x + y + z)(r_ix + s_iy + t_iz)$. Then $\operatorname{pow}_{\Gamma_i} P = \Gamma_i(P)$. In particular, $\operatorname{pow}_{\Gamma_i} A = \Gamma_i(1, 0, 0) = r_i$ and, similarly, $\operatorname{pow}_{\Gamma_i} B = s_i$ and $\operatorname{pow}_{\Gamma_i} C = t_i$.
Finally, the radical axis is

$$\begin{aligned} & \operatorname{pow}_{\Gamma_1} P = \operatorname{pow}_{\Gamma_2} P \\ \iff & \Gamma_1(x, y, z) = \Gamma_2(x, y, z) \\ \iff & r_1x + s_1y + t_1z = r_2x + s_2y + t_2z \\ \iff & (\operatorname{pow}_{\Gamma_1} A - \operatorname{pow}_{\Gamma_2} A)x + (\operatorname{pow}_{\Gamma_1} B - \operatorname{pow}_{\Gamma_2} B)y + (\operatorname{pow}_{\Gamma_1} C - \operatorname{pow}_{\Gamma_2} C)z = 0. \end{aligned}$$

We still use the Miquel point E of $ABCD$. Notice that the problem is equivalent to proving that lines MP, NQ, and EK are concurrent. The main idea is writing these three lines as radical axes. In fact, by definition of points M, N, and E:

- MP is the radical axis of the circumcircles of PAS and PBR;
- NQ is the radical axis of the circumcircles of QCR and QDS;
- EK is the radical axis of the circumcircles of KBC and KAD.

Looking at these facts and the diagram, it makes sense to take triangle KQP the reference triangle. Because of that, we do not really need to draw circles nor even points M and N, as all powers can be computed directly from points in lines KP, KQ, and PQ.



Associate P with the x-coordinate, Q with the y-coordinate, and K with the z-coordinate. Applying the lemma, the equations of lines PM, QN, and EK are

- MP: $(KA \cdot KP - KB \cdot KP)x + (QS \cdot QP - QR \cdot QP)y = 0$
- NQ: $(KC \cdot KQ - KD \cdot KQ)x + (PR \cdot PQ - PS \cdot PQ)z = 0$
- MP: $(-QC \cdot QK + QD \cdot QK)y + (PB \cdot PK - PA \cdot PK)z = 0$

These equations simplify to

- MP: $(AB \cdot KP)x + (PQ \cdot RS)y = 0$
- NQ: $(-CD \cdot KQ)x + (PQ \cdot RS)z = 0$
- MP: $(CD \cdot KQ)y + (AB \cdot KP)z = 0$

Now, if $u = AB \cdot KP$, $v = PQ \cdot RS$, and $w = CD \cdot KQ$, it suffices to show that

$$\begin{vmatrix} u & v & 0 \\ -w & 0 & v \\ 0 & w & u \end{vmatrix} = 0,$$

which is a straightforward computation.